Quantum Theory Derived from Logic

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ABSTRACT

After a very brief introduction to logic, I show how a type of path integral can be constructed in terms of propositional logic. This logic is then transformed into the Feynman Path Integral of quantum mechanics using general techniques.

Reality can be defined as the conjunction of all the facts we observe. Even space itself consists of a collection of points all in conjunction with each other. And we describe each point as having individual coordinates. A conjunction of points, however, means that every point in fact logically implies every other point. And it will be shown that an implication between two points equates to the disjunction of every possible path from one point to another. Each path consists of a conjunction of implications, the first point implying the second, in conjunction with the second point implying the third, in conjunction with the third implying the forth, etc. Implication is then represented in set theory using subsets, if a set exists, then its subset exists. And the inclusion of a subset can be represented mathematically using the Dirac measure, which equals 1 if the subset is included and is 0 otherwise. This can be manipulated into the Kronecker delta, δ_{ij}, which is 1 if $i = j$ and is 0 if $i \neq j$. With implication represented by the Kronecker delta, it is straightforward to show that disjunction is represented by addition, and that conjunction is represented by multiplication. The disjunction of paths then has a mathematical representation. In the case of a continuous space, the Kronecker delta is replaced with the Dirac delta function. When the exponential Gaussian function is used to represent the Dirac delta function, the conjunction of implications for a path become the multiplication of exponential functions. The exponents then add up to form an Action integral, and the disjunction of every possible path forms the Feynman Path Integral of quantum mechanics. This is 1st quantization. The wave function is the mathematical representation of logical implication. This process can be iterated to give us the quantum field theory of 2nd quantization. And the process can be iterated again to even give us 3rd quantization if needed. I also show where the Born Rule comes from to give us probabilities from the square modulus of the wave function. And finally, I give some reason to expect that these iterations prescribe that the complex numbers iterate to quaternions and then to octonions, which are believed to be responsible for the U(1)×SU(2)×SU(3) symmetry of the Standard Model.

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INTRODUCTION

Historically, quantum mechanics was developed in a rather ad-hoc manner, using trial and error to find some mathematics that eventually proved useful in making predictions. But the ultimate reasons why nature operates according to the equations of quantum mechanics has remained elusive. And some students of physics are mystified to the point of frustrations by quantum mechanics because there does not seem to be any underlying principle that justifies it. Where does the wave-function come from? How can the imaginary square-root of a probability have anything to do with reality? Some complain that it is counter-intuitive and even illogical. But the goal of this article is to prove that quantum mechanics can be derived from classical logic without any physical assumptions.

Those most interested in foundational issues are those exposed to the subject for the first time. It's usually easier to accept more complicated implications of a theory when the basic premises of it are well understood. Therefore, in order to broaden the audience, I include about a page worth of paragraphs briefly describing the basic introductory definitions in logic. And I include about a page worth of introduction to the integration process of calculus. The fundamentals in a subject should be relatively easy, so my intention is to keep this article under a sophomore college level. It is hoped that the ease of this material will be appreciated. The web pages I link to should contain a bibliography for those interested in further reading. The math is relatively easy but can be a bit tedious; I wanted to be as complete as possible. The article on the website (logictophysics.com/QMlogic.html) provides features to aid in traversing the details. Advanced readers can skip to the next section if they are familiar with the symbols used in logic.

And so to start… Anyone can make claims about any subject they like, but that only brings up questions as to what evidence there is to support those claims and what those claims imply. And some may like to think they are being reasonable in what they believe. But how can we know that the conclusions they reach are correctly derived in a reasonable way? Logic is the study of correct argumentation. Given facts in relation to each other, logic is a tool to help us determine what other truths these facts equate to or imply. In this section I briefly touch on three topics in logic: propositional logic, set theory, and predicate logic.

Propositional logic studies how the truth or falsity of statements effect the truth or falsity of other statements. Propositions are the same thing as statements or facts or claims which can either be true or they can be false, but they cannot be neither true nor false, and they cannot be both true and false at the same time. Propositional logic does not consider what the statements are about; it does not consider whether the statements are about abstract concepts such as math, or about physical facts or about feeling, emotions, or beauty. All propositional logic does is label different statements with different letters such as ***a, b, c***, etc. and treats them as variables whose values can be either true or false. Then the formulas of propositional logic can be applied to any subject and form the basis of valid reasoning about it. I will use ***T*** for true and ***F*** for False.

Compound statements can be constructed from simple statements using connectives such as AND and OR and IMPLIES and NOT. And the truth of the compound statement depends on how the simple statements are connected. Symbols are used for these connectives. I will use $\wedge$ for AND (= conjunction), $\vee$ for OR (= disjunction), and $\rightarrow$ for IMPLIES (= material implication), and $\neg$ for NOT (= negation). Below is a truth-table that shows the effect of these connectives on two statements, ***p*** and ***q***.

1	2	3	4	5
p q	¬ ***p***	***p***∧***q***	***p***∨***q***	***p*** ⟶ ***q***
F F	***T***	***F***	***F***	***T***
F T	***T***	***F***	***T***	***T***
T F	***F***	***F***	***T***	***F***
T T	***F***	***T***	***T***	***T***

Column 1 in the table lists every possible combination of ***T*** and ***F*** that ***p*** and ***q*** can have. Column 2 shows that the operation of negation (NOT) has the effect of reversing the truth-value of ***p***. If ***p*** is ***T***, then ¬ ***p*** is ***F***, and visa versa. Column 3 shows that the statement "***p*** AND ***q***" is ***T*** only when both ***p*** is ***T*** and ***q*** is ***T***. Column 4 shows that "***p*** OR ***q***" is ***T*** whenever either ***p*** is ***T*** or ***q*** is ***T*** or when both are ***T***. Material implication is the IF, THEN function of logic. If ***p*** implies ***q***, this means if ***p*** is true, then ***q*** is true. To say that ***p*** implies ***q*** is the same thing as saying if ***p*** then ***q***, or ***p*** proves ***q***, or ***p*** therefore ***q***, or ***p*** results in ***q***, or ***p*** causes ***q***, etc. Here the first operand, ***p***, is called the premise, and the second operand, ***q***, is called the consequence. Column 5 shows the relationship of material implication. It is true that ***p*** proves ***q*** for any truth-values of ***p*** and ***q*** except when ***p*** is ***T*** but ***q*** is ***F***. The consequences might still be true regardless of the premises on which they're based, but you cannot have premises that are true without consequences; that would mean there is not an implication between them. Some things to note are that conjunction (AND) is commutative, which means you can reverse the order of the operands, ***p*** and ***q***, so you get $p \wedge q = q \wedge p$. It is also true that disjunction (OR) is commutative. But implication (→) is not commutative, $p \rightarrow q$ is not equal to $q \rightarrow p$.

You can find a more complete video lecture series on basic logic here, (https://www.youtube.com/watch?v=aiS2EgYxtYc).

There is an on-line service that provides a truth-table generator here, (http://web.stanford.edu/class/cs103/tools/truth-table-tool/). If you want to gain more confidence in these logic statements, simply enter the statement in the box, and a truth-table will appear. However, text characters must be entered for logic symbols. Use "/\" for AND, "\/" for OR, "=>" for IMPLIES, "~" for NOT, and "<=>" for EQUALS. For example, for the AND statement enter the following text into the box (without the quotation marks):
"p /\ q". For some of the logic expressions written below, a text version is provided that you can cut and paste into this truth-table generator. Set theory constructs lists of objects called elements. For example, the set S whose elements are objects labeled a and b and c and d is written as S = {a,b,c,d}, where a ∈ S symbolizes that a is an element of the set S. Then set theory examines how differing sets can be combined. You can combine sets by considering the union of sets, or the intersection between them or the complement of a set. For example, if you have only two sets, A = {a,b,c,d,e,f} and B = {d,e,f,g,h}, then the union between them is
A ∪ B = {a,b,c,d,e,f,g,h}. The intersection of the two sets is A ∩ B={d,e,f}. And if these two sets contain all the possible elements in the universe of our discourse, then the complement of B is B={a,b,c}. It is also possible to have sets which are subsets of other sets. For example, the set C={a,c,e} is a subset of A, symbolized as C ⊂ A or as A ⊃ C which says C is a subset of A, same thing as saying A is a superset of C.

Many times propositions can be described as objects with a particular property. In predicate logic, if a specific object labeled q has the property labeled **P**, then **P**q is the notation for saying it is true that q has the property **P**. The extension of a predicate, **P**, here labeled P, is the set of all those specific objects which have the property **P**. In other words, if P={q_1,q_2,q_3,q_4}, where it is true that **P**q_1 and **P**q_2 and **P**q_3 and **P**q_4. The expansion of the predicate **P** is a proposition, here labeled ***P***, which is the conjunction of statements of all those objects that have the property **P**. In symbols, ***P*** = **P**q_1 ∧ **P**q_2 ∧ **P**q_3 ∧ **P**q_4. If it is understood that $\boldsymbol{q_1}$, $\boldsymbol{q_2}$, $\boldsymbol{q_3}$, and $\boldsymbol{q_4}$ are each propositions such that $\boldsymbol{q_1}$=**P**q_1, $\boldsymbol{q_2}$=**P**q_2, $\boldsymbol{q_3}$=**P**q_3, and $\boldsymbol{q_4}$=**P**q_4, then we can shorten the notation to $\boldsymbol{P} = \boldsymbol{q_1} \wedge \boldsymbol{q_2} \wedge \boldsymbol{q_3} \wedge \boldsymbol{q_4}$. And we can consider the consistency between all the statements in the set.

THE PATHS OF LOGIC

Consistency among statements in a theory means that no statement in the theory can prove to be both true and false. And this means, of course, that no statement in the theory will prove itself false. So if we are given a set of statements that are asserted to be true, then consistency requires that no statement in the set will ever prove false any other statement in that set. Or in symbols, if $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ are asserted to coexist as true statements of the theory, then

$$\neg (\boldsymbol{q_1} \rightarrow \neg \boldsymbol{q_2}) \tag{1}$$

Put ~($\boldsymbol{q_1}$ => ~$\boldsymbol{q_2}$) in the truth-table generator.

But it should be noted that

$$\neg (\boldsymbol{q_1} \rightarrow \neg \boldsymbol{q_2}) = \boldsymbol{q_1} \wedge \boldsymbol{q_2} \tag{2}$$

Put ~($\boldsymbol{q_1}$ => ~$\boldsymbol{q_2}$) <=> ($\boldsymbol{q_1}$ ∧ $\boldsymbol{q_2}$) in the truth-table generator. Notice that the result is true for all values of $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$. This means that it is a valid argument in all circumstances. It is sometimes called a tautology.

And if Equation [2] is true between any two statements in the set, then a consistent set can be seen as the conjunction of all its statements:

$$\boldsymbol{q_1} \wedge \boldsymbol{q_2} \wedge \boldsymbol{q_3} \wedge \ldots \wedge \boldsymbol{q_n} = \wedge_{i=1}^{n} \boldsymbol{q_i} \tag{3}$$

where all the $\boldsymbol{q_i}$ belong to the same set, and where n could be infinite, and where the $\wedge^{n}{}_{i=1}$ symbol used here is the logical conjunction of n statements.

To apply these ideas to nature, we can say that reality consists of all the objects within it. We can use the letter **U** to symbolize the property of belonging to the universe, and symbols such as q_1, q_2,q_3, q_4, etc. to represent various kinds of objects. We write **U**q_1, **U**q_2, **U**q_3, etc. to represent the statements that those objects have the property of actually existing in the universe. We can abbreviate those statements as $\boldsymbol{q_1}$, $\boldsymbol{q_2}$, $\boldsymbol{q_3}$, etc., which means $\boldsymbol{q_1}$= **U**q_1, $\boldsymbol{q_2}$=**U**q_2, $\boldsymbol{q_3}$=**U**q_3, etc., and they describe facts in the universe in terms of propositions that can be considered either true or false. The extension of the property **U** would be the set U={q_1, q_2, q_3,...}, and the expansion of **U** would be the proposition $\boldsymbol{U}$=$\boldsymbol{q_1} \wedge \boldsymbol{q_2} \wedge \boldsymbol{q_3} \wedge$... And we would say that the universe consists of all the facts in reality coexisting in conjunction with each other.

It may be that some of the facts, $\boldsymbol{q}_i$, might be broken down into a conjunction of even more propositions which represent even smaller objects that have differing properties. And it may be that still other facts, $\boldsymbol{q}_j$, may share some of these differing objects in common. But it's still clear that the extension of these differing objects are subsets of the universal set, U. And the expansion of these objects only contribute propositions that exist in conjunction with everything else. So we can ultimately describe the universe as consisting of a conjunction of all the facts that describe all the parts of the universe. We use propositions to describe individual facts in reality all the time. For we describe situations in nature with propositions - this physical situation has this or that property, it's made of these parts, it's located at this place at this time. And we often argue about whether a statement about reality is actually true. We use the word "true" for those propositions that do describe what's real and "false" for those propositions that do not describe what's real. Larger physical systems are described with smaller physical subsystems. And we strive to find the smallest constituents of reality which will themselves always end up being described with one statement or another that we call true.

So nature can be considered to be a consistent set of statements. And we expect that no fact in reality will ever contradict any other fact in reality. Just looking around we see that the chair we are sitting on exists AND the floor holding up the chair exists AND the computer screen we are reading exists AND the room we are in exists AND the walls exist AND the doors of the room exist AND the atoms they are made of exist, etc., etc., ad infinitum. We presume this coexistence between facts at every level of existence down to the most microscopic level even though it is not observable with our eyes. For if this much were not true, I don't suppose we would be able to describe anything in reality. So in the most general sense, I think it's fair to describe reality at the smallest level as consisting of a consistent set of propositions. That isn't to say we know what all the facts are or what properties they have, but whatever laws of physics there are, we suppose they come from some sort of underlying consistency.

Continuing from Equation [3] , it should be realized that

$$\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \longrightarrow (\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_2 \longrightarrow \boldsymbol{q}_1) \qquad [4]$$

Put $(\boldsymbol{q}_1 \wedge \boldsymbol{q}_2) \Rightarrow ((\boldsymbol{q}_1 \Rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_1))$ in the truth-table generator. Notice that it is always true.

So what this means for the whole conjunction of reality is

$$\wedge_{i=1}^{n} \boldsymbol{q}_i \longrightarrow \wedge_{i=1}^{n} \wedge_{j=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \wedge_{ij=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) \qquad [5]$$

This conjunction would include factors such as $(\boldsymbol{q}_i \rightarrow \boldsymbol{q}_i)$ which are true by the definition of material implication. And such factors do not change the conjunction since $\boldsymbol{p} = \boldsymbol{p} \wedge \boldsymbol{T}$ for any proposition $\boldsymbol{p}$. You can always factor in a truth in a conjunction.

For example, if both i and j run from 1 to 4 in Equation [5], then the following conjunction is obtained, and you can put the following equation in the truth-table generator:

$$\begin{aligned}
&(\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \wedge \boldsymbol{q}_3 \wedge \boldsymbol{q}_4) \Rightarrow \\
&\quad (\boldsymbol{q}_1 \Rightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_1 \Rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_1 \Rightarrow \boldsymbol{q}_3) \wedge (\boldsymbol{q}_1 \Rightarrow \boldsymbol{q}_4) \\
&\wedge (\boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_3) \wedge (\boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_4) \\
&\wedge (\boldsymbol{q}_3 \Rightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_3 \Rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_3 \Rightarrow \boldsymbol{q}_3) \wedge (\boldsymbol{q}_3 \Rightarrow \boldsymbol{q}_4) \\
&\wedge (\boldsymbol{q}_4 \Rightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_4 \Rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_4 \Rightarrow \boldsymbol{q}_3) \wedge (\boldsymbol{q}_4 \Rightarrow \boldsymbol{q}_4)
\end{aligned}$$

Notice that it is always true. Also, notice that parenthesis are inserted in the box between conjunctions to show which conjunction is evaluated first. This does not interfere with the calculation since $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d} = \boldsymbol{a} \wedge (\boldsymbol{b} \wedge (\boldsymbol{c} \wedge \boldsymbol{d}))$.

The conjunction on the left hand side (*LHS*) of Equation [5], only implies the right hand side (*RHS*); it is not an equivalence. When all the $\boldsymbol{q}_i$ are $\boldsymbol{T}$, the *LHS* equals the *RHS*, and both sides are $\boldsymbol{T}$. If there is a mixture of $\boldsymbol{T}$ and $\boldsymbol{F}$ for the $\boldsymbol{q}_i$, then the *LHS* will be $\boldsymbol{F}$ since there is an $\boldsymbol{F}$ in a conjunction. But on the *RHS*, there will be factors of the form $(\boldsymbol{F} \rightarrow \boldsymbol{T}) = \boldsymbol{T}$, and when those same factors are reversed elsewhere in the conjunction, there will be factors of the form $(\boldsymbol{T} \rightarrow \boldsymbol{F}) = \boldsymbol{F}$, making the conjunction on the *RHS* false just as it is on the *LHS*. The only difference between the *LHS* and the *RHS* is when all the $\boldsymbol{q}_i$ are $\boldsymbol{F}$. Though the conjunction on the *LHS* is false when all $\boldsymbol{q}_i$ are false, all the implications on the *RHS* are $\boldsymbol{T}$ when all the $\boldsymbol{q}_i$ are $\boldsymbol{F}$. This is because $(\boldsymbol{F} \rightarrow \boldsymbol{F}) = \boldsymbol{T}$ is a true statement by definition of implication. Yet, if it is safe to at least assume that something in the set is true, then Equation [5] becomes an effective equality. For then there will be an implication somewhere on the *RHS* of the form $(\boldsymbol{T} \rightarrow \boldsymbol{F}) = \boldsymbol{F}$, which would make the conjunction on the *RHS* false just as the *LHS* would be. And in the case of reality, it's probably safe to assume that there must be something that truly exists. For we can at least say that the universe exists. (Try changing the first => to the equal sign, <=>, for the last equation inserted in the generator.)

So how are paths constructed? Consider the following:

$$((\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1) = ((\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_0) \wedge (\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1)) \vee ((\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_1)),$$

where $\boldsymbol{q}_0$ is the start of the path, and $\boldsymbol{q}_1$ is the end of the path. This is obvious, because both $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_0)$ and $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_1)$ are true, and we have $\boldsymbol{q} \wedge \boldsymbol{T} = \boldsymbol{q}$. So we are left with $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1) \vee (\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1)$, but this is just $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1)$ since $\boldsymbol{q} \vee \boldsymbol{q} = \boldsymbol{q}$.

To check this, put ($\boldsymbol{q}_0$ => $\boldsymbol{q}_1$) <=> (($\boldsymbol{q}_0$ => $\boldsymbol{q}_0$) ∧ ($\boldsymbol{q}_0$ => $\boldsymbol{q}_1$)) ∨ (($\boldsymbol{q}_0$ => $\boldsymbol{q}_1$) ∧ ($\boldsymbol{q}_1$ => $\boldsymbol{q}_1$))

in the truth-table generator. Notice that it is always true. This can also be written as,

$(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_1) = \vee^{1}_{j=0} (\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_j) \wedge (\boldsymbol{q}_j \rightarrow \boldsymbol{q}_1)$, where $\vee^{1}_{j=0}$ is the disjunction of two terms.

Here I'm comparing an implication from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$ to a step in a path, for in some sense the premise leads us to the conclusion like a step from one place to another. The last equation represents a very short, one step path from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$, but we can insert intermediate steps. Let the start of the path be $\boldsymbol{q}_0$, and the end of the path be $\boldsymbol{q}_2$. And now let's insert an intermediate step, $\boldsymbol{q}_1$, between them. This is now a two-step path, and the paths from $\boldsymbol{q}_0$ to $\boldsymbol{q}_2$ would give us,

$$((\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_2) = ((\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_0) \wedge (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_2)) \vee ((\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_2)) \vee ((\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_2 \longrightarrow \boldsymbol{q}_2)).$$

To check this, put

($\boldsymbol{q}_0$ => $\boldsymbol{q}_2$) <=> (($\boldsymbol{q}_0$ => $\boldsymbol{q}_0$) ∧ ($\boldsymbol{q}_0$ => $\boldsymbol{q}_2$)) ∨ (($\boldsymbol{q}_0$ => $\boldsymbol{q}_1$) ∧ ($\boldsymbol{q}_1$ => $\boldsymbol{q}_2$)) ∨ (($\boldsymbol{q}_0$ => $\boldsymbol{q}_2$) ∧ ($\boldsymbol{q}_2$ => $\boldsymbol{q}_2$))

in the generator. Notice that it is always true.

And it can also be written as, $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_2) = \vee^{2}_{j=0} (\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_j) \wedge (\boldsymbol{q}_j \rightarrow \boldsymbol{q}_2)$.

This can be generalized to any number of two-step paths:

$$(\boldsymbol{q_0} \longrightarrow \boldsymbol{q_n}) = \vee_{j=0}^{n} (\boldsymbol{q_0} \longrightarrow \boldsymbol{q_j}) \wedge (\boldsymbol{q_j} \longrightarrow \boldsymbol{q_n}), \qquad [6]$$

To check this for a *j* that runs from 0 to 4, insert the following in the generator:

$(\boldsymbol{q_0} \Rightarrow \boldsymbol{q_4}) \Leftrightarrow$
$((\boldsymbol{q_0} \Rightarrow \boldsymbol{q_0}) \wedge (\boldsymbol{q_0} \Rightarrow \boldsymbol{q_4})) \vee ((\boldsymbol{q_0} \Rightarrow \boldsymbol{q_1}) \wedge (\boldsymbol{q_1} \Rightarrow \boldsymbol{q_4}))$
$\vee ((\boldsymbol{q_0} \Rightarrow \boldsymbol{q_2}) \wedge (\boldsymbol{q_2} \Rightarrow \boldsymbol{q_4})) \wedge ((\boldsymbol{q_0} \Rightarrow \boldsymbol{q_3}) \wedge (\boldsymbol{q_3} \Rightarrow \boldsymbol{q_4}))$
$\vee ((\boldsymbol{q_0} \Rightarrow \boldsymbol{q_4}) \wedge (\boldsymbol{q_4} \Rightarrow \boldsymbol{q_4}))$

Notice that this is always true.

There's no value of $\boldsymbol{q_j}$, $\boldsymbol{T}$ or $\boldsymbol{F}$, that can negate the equality of Equation [6]. To prove this, there are two cases to consider: either case 1: $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = \boldsymbol{F}$, or case 2: $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = \boldsymbol{T}$.

In case 1 , $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = \boldsymbol{F}$, this can only happen if $\boldsymbol{q_0} = \boldsymbol{T}$ and $\boldsymbol{q_n} = \boldsymbol{F}$.

Then for a $\boldsymbol{q_j} = \boldsymbol{T}$, $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_j})$ will be equal to $(\boldsymbol{T} \rightarrow \boldsymbol{T})$, which is true.
However, $(\boldsymbol{q_j} \rightarrow \boldsymbol{q_n})$ will be $(\boldsymbol{T} \rightarrow \boldsymbol{F})$, which is false. And $\boldsymbol{T} \wedge \boldsymbol{F} = \boldsymbol{F}$. So that term in the disjunction will be false.

But for a $\boldsymbol{q_j} = \boldsymbol{F}$, $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_j})$ will be equal to $(\boldsymbol{T} \rightarrow \boldsymbol{F})$, which is false. So again that term in the disjunction will be false.
So if $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = \boldsymbol{F}$, then all the terms will be false no matter the value of $\boldsymbol{q_j}$, and both sides of Equation [6] will be false.

In case 2, $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = \boldsymbol{T}$, this can happen in three ways.
Way 1: $\boldsymbol{q_0} = \boldsymbol{F}$ and $\boldsymbol{q_n} = \boldsymbol{T}$,
Way 2: $\boldsymbol{q_0} = \boldsymbol{F}$ and $\boldsymbol{q_n} = \boldsymbol{F}$,
Way 3: $\boldsymbol{q_0} = \boldsymbol{T}$ and $\boldsymbol{q_n} = \boldsymbol{T}$.

For Way 1, $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_j})$ will be $(\boldsymbol{F} \rightarrow \boldsymbol{q_j})$, which is true for any $\boldsymbol{q_j}$. And $(\boldsymbol{q_j} \rightarrow \boldsymbol{q_n})$ will be $(\boldsymbol{q_j} \rightarrow \boldsymbol{T})$, which is also true for any $\boldsymbol{q_j}$. So all the disjunction terms will be true as well. So both sides of Equation [6] will be true.

For Way 2 , there will be at least one term that is true, namely, when $j=0$. For $\boldsymbol{q_0}$ has already been assigned to be false. Then $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_j})$ will be $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_0})$, which is true, and $(\boldsymbol{q_j} \rightarrow \boldsymbol{q_n})$ will be $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = (\boldsymbol{F} \rightarrow \boldsymbol{q_n})$, which is also true. This will make the $j=0$ term true, which makes the whole disjunction true.

For Way 3 , there will be at least one term that is true, namely, when $j=n$. For $\boldsymbol{q_n}$ has already been assigned to be true. Then $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_j})$ will be $(\boldsymbol{q_0} \rightarrow \boldsymbol{q_n}) = (\boldsymbol{q_j} \rightarrow \boldsymbol{T})$, which is true, and $(\boldsymbol{q_j} \rightarrow \boldsymbol{q_n})$ will be $(\boldsymbol{q_n} \rightarrow \boldsymbol{q_n})$, which is also true. This will make the $j=n$ term true, which makes the whole disjunction true.

So there is no value of $\boldsymbol{q_j}$ that can make the equality in Equation [6] a false statement.

Equation [6] represents n parallel paths of two steps each. The index j cycles through all n propositions in the universal set so that $\boldsymbol{q}_j$ acts like a variable taking the place of various propositions. For each value of j, $\boldsymbol{q}_j$ represents a different proposition. Note that since $\boldsymbol{q}_j$ is the only variable in Equation [6]. The factors $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_j)$ and $(\boldsymbol{q}_j \rightarrow \boldsymbol{q}_n)$ can be thought of as functions of the single variable $\boldsymbol{q}_j$, with $\boldsymbol{q}_0$ and $\boldsymbol{q}_n$ being held constant. Then Equation [6] can be thought of as a type of mathematical expansion in terms of other functions.

Equation [6] can be iterated to give all possible paths of 3 steps each. For example, let $(\boldsymbol{q}_0 \rightarrow \boldsymbol{q}_j)$ in Equation [6] be,

$$(\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_j) = \vee_{i=0}^{n} (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_i) \wedge (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j).$$

And insert this into Equation [6] to get,

$$(\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_n) = \vee_{j=0}^{n} [\vee_{i=0}^{n} (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_i) \wedge (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j)] \wedge (\boldsymbol{q}_j \longrightarrow \boldsymbol{q}_n),$$

which can be written as,

$$(\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_n) = \vee_{i,j=0}^{n} (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_i) \wedge (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) \wedge (\boldsymbol{q}_j \longrightarrow \boldsymbol{q}_n).$$

And we can iterate this m number of times to get,

$$(\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_n) = \vee_{i_1,i_2,i_3,i_4,\ldots,i_m=0}^{n} (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_{i_1}) \wedge (\boldsymbol{q}_{i_1} \longrightarrow \boldsymbol{q}_{i_2}) \wedge (\boldsymbol{q}_{i_2} \longrightarrow \boldsymbol{q}_{i_3}) \wedge (\boldsymbol{q}_{i_3} \longrightarrow \boldsymbol{q}_{i_4}) \wedge \ldots \wedge (\boldsymbol{q}_{i_m} \longrightarrow \boldsymbol{q}_n). \qquad [7]$$

Each term in this disjunction is a path of m steps. But this also includes terms like,

$$(\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_0) \wedge (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_0) \wedge (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_0) \wedge (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_0) \wedge \ldots \wedge (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_n) = (\boldsymbol{q}_0 \longrightarrow \boldsymbol{q}_n),$$

which is a 1 step path. So Equation [7] contains every possible path, including paths of 1 step, 2 steps, 3 steps, up to m steps each.

If you want to check Equation [7], the page here, (logictophysics.com/4StepPaths.html), gives the logical expression for Equation [7] when $n=4$ and $m=3$. You may cut and paste this into the truth-table generator. And you may experiment with changing m.

Factors like $(\boldsymbol{q}_{i_1} \rightarrow \boldsymbol{q}_{i_2})$ in Equation [7] are functions of two variables, since both $\boldsymbol{q}_{i_1}$ and $\boldsymbol{q}_{i_2}$ act like variables which cycle through various propositions. If $m=n$, so that the i's range through every possible state in the universal set, then Equation [7] is the combination of every possible path through the universal set. Already we can see this is setting us up to derive Feynman's Path Integral. All we need to do is find a means to map these logical operation to mathematical operations. The next section is an effort to do just that.

It might be interesting to consider that Equation [7] could have been anticipated long ago. For it seems to represent every disagreement we have. We might agree about the state of affairs at some point in the past, and we might agree about some other point after that. But we might disagree about what sequence of events got us from the first point to the second point. One party proposes one sequence of event. The other party proposes a different sequence of events. And we are left considering the alternative sequences of events. For example, a man on trial for murder, both parties agree that the victim was alive at some point and then was found dead at another point. Prosecution will argue that a series of events happened to prove that the accused committed the crime. Whereas, the Defense will argue a different sequence of events in which the man is innocent. The jury ends up considering alternative sequences of events.

THE MEASURE OF IMPLICATION

Propositional logic uses connectives like AND and OR and NOT that operate on statements that are true or false. But physics is expressed in terms of mathematical operations that act on numerical values. So if we wish to go from logic to math, we need a way to assign mathematical operations to logical connectives and to give numerical value to propositional statements.

It seems that mathematics itself is built on the foundation of set theory; Zermelo–Fraenkel, (en.wikipedia.org/wiki/Zermelo–Fraenkel_set_theory), set theory serves as the most common foundation of mathematics. And since we are trying to find a mathematical expression for Equation [7] that relies heavily on implication, we must ask if there is a set-theoretic representation of implication. It turns out that there is.

Implication is expressed in set theory using subsets as shown here, (en.wikipedia.org/wiki/Material_conditional). If a set exists, then any of its subsets exist. If the set A is defined to be {a,b,c,d,e,f}, and set B is defined to be {c,d,e}, then if A exists, B exists. This can be written as $A \supset B$, which means A is a superset of B. It is the same as writing $B \subset A$, which means B is a subset of A. This has the same truth-table as the material implication of propositional logic. If A exists, then B exists. But if B exists (or is defined), this does not mean that A exists. And if A does not exist, then B cannot exist. What you cannot have is that A exists but B does not; that would deny the definition of a subset. So we have a set-theoretical definition of implication. But is there a numeric function for this set theoretical implication?

Recall that a proposition, ***q***, can be written as **P**q, which is true if the object q has the property **P** and is false if q does not have the property **P**. But the property **P** has the extension $P=\{q_1,q_2,q_3,...\}$, the set of all such objects that have that property. This means that ***q*** is true if $(q \in P)$ and ***q*** is false if $(q \notin P)$. So we have propositions and implications in terms of set theory. But how does this relate to numbers?

Numbers are defined as the count of how many elements are in a set; this is called the cardinality of a set. Zero is the cardinality of the empty set, one is defined as the set with one element, and two is defined as the set with two elements in it, etc. If P is the extension of a property to which only q may belong, then $(q \notin P)$ mean that P is empty and has the cardinality of zero, and $(q \in P)$ means that P has the cardinality of one. So we seek a function which gives 1 for set inclusion and 0 otherwise.

The Dirac measure accomplishes this, as shown here, (en.wikipedia.org/wiki/Dirac_measure). The Dirac measure is denoted $\boldsymbol{\delta}_x(A)$ and is defined such that,

$$\boldsymbol{\delta}_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

where x is a possible element of some arbitrary set, A.

In the language here, if the proposition $\boldsymbol{x} = (x \in A)$ is true, then the Dirac measure maps ***x*** to the value of 1. And if ***x*** is a false proposition (or $x \notin A$), then the Dirac measure maps ***x*** to the value of 0. So the Dirac measure maps ***T*** to 1 and ***F*** to 0. And this makes intuitive sense. For if we ask whether something exists, then if the answer is “yes, that's true”, we have at least 1 sample of that thing. But if the answer is “no, that's false”, then we have zero samples of that thing.

But if the set A has been defined to be the set {a,b,c,d,x,e,f,g,h}, then we can take the expansion of it to get the proposition, $\boldsymbol{A} = \boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}\wedge\boldsymbol{d}\wedge\boldsymbol{x}\wedge\boldsymbol{e}\wedge\boldsymbol{f}\wedge\boldsymbol{g}\wedge\boldsymbol{h}$, where $\boldsymbol{a} = (a \in A)$, $\boldsymbol{b} = (b \in A)$, etc. And we can consider the truth and falsity of the propositions $\boldsymbol{A}$ and $\boldsymbol{x}$ independently from one another and ask how the truth-value of each are related. If $\boldsymbol{A}$ is true, then so is $\boldsymbol{x}$. But if $\boldsymbol{A}$ is false, this does not mean that $\boldsymbol{x}$ is false. It may be that any of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$, or $\boldsymbol{h}$ is false, which means the proposition $\boldsymbol{A}$ may include a conjunction with the statement, $\boldsymbol{g}$, but the set A does not include the element, g. However, if $\boldsymbol{x}$ is false, then so must $\boldsymbol{A}$ be false. We cannot have that $\boldsymbol{x}$ is false and $\boldsymbol{A}$ is true. This mimics the truth-table of implication.

So far, we found that the inclusion or not of a subset within a set expresses logical implication. And we found that the Dirac measure gives a numeric value depending on whether a specific element is included or not in a set. We can recognize, however, that an element can be seen as a set of that single element. For it's always true that $(x \in \{x\})$. Then the Dirac measure is a numeric representation of set inclusion, which is a set-theoretic representation of material implication. So we have a numeric function from logic to math.

But the situation we have is that all the propositions in Equation [7] each describe the inclusion or not of a corresponding element in a universal set. In other words, all the propositions in Equation [7] represent elements, none are described as sets that contain other elements. So the question is how do we use the Dirac measure to represent implication between elements?

In the notation for the Daric measure, $\boldsymbol{\delta}_x(A)$, notice that x is an element and A is a set and not an element. Yet, we need a math representation for the implication between one element and another element. For paths were constructed in Equation [7] using the implication between propositions, where each proposition describes a single element in the universal set. So we need to manipulate $\boldsymbol{\delta}_x(A)$ to be more of the form $\boldsymbol{\delta}_x(\{y\})$, which would mathematically represent more closely the implication between two propositions.

To accomplish this, note that the set A in the notation of $\boldsymbol{\delta}_x(A)$ is a set whose number of elements is not specified. So we should still have $\boldsymbol{\delta}_x(A)$ representing implication even if A is shrunk down to the size of an element. Let A shrink down to an element, call it y. Then, in that case, we have A= {y}, and we can write,

$$\boldsymbol{\delta}_{\mathrm{x}}(\mathrm{A}) = \boldsymbol{\delta}_{\mathrm{x}}(\{\mathrm{y}\}) = \boldsymbol{\delta}_{\mathrm{yx}} = \begin{matrix} 1 \text{ if } \mathrm{x} = \mathrm{y}, \\ \\ 0 \text{ if } \mathrm{x} \neq y. \end{matrix}$$

For we know that if the element x is the element y, or x= y, then $x \in \{y\}$ would equal $x \in \{x\}$, which is inherently true and gets mapped to 1. Otherwise, if $x \neq y$, then $x \in \overline{\{x\}}$, where $\overline{\{x\}}$ is an element other than x. And we know that $x \in \overline{\{x\}}$ is an inherently false statement that gets mapped to 0.

Previously when we considered A = {a,b,c,d,x,e,f,g,h}, the expansion was $\boldsymbol{A} = \boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}\wedge\boldsymbol{d}\wedge\boldsymbol{x}\wedge\boldsymbol{e}\wedge\boldsymbol{f}\wedge\boldsymbol{g}\wedge\boldsymbol{h}$. But now, when we think of A={y}, the expansion is $\boldsymbol{A} = \boldsymbol{y}$. So $\boldsymbol{A} \rightarrow \boldsymbol{x}$ becomes $\boldsymbol{y} \rightarrow \boldsymbol{x}$, and $\boldsymbol{\delta}_x(\{y\})$ is a mathematical representation of $\boldsymbol{y} \rightarrow \boldsymbol{x}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ each refer to an element, which is what is needed for the conjunction of implications in Equation [7]. And it's appropriate that implication should be mapped to a function with numeric valued, namely $\boldsymbol{\delta}_x(\{y\})$. For the implication, $\boldsymbol{y} \rightarrow \boldsymbol{x}$, is also a statement in and of itself that evaluates to $\boldsymbol{T}$ or $\boldsymbol{F}$, depending on whether $\boldsymbol{x}$ or $\boldsymbol{y}$ is $\boldsymbol{T}$ or $\boldsymbol{F}$. So its representation, $\boldsymbol{\delta}_x(\{y\})$, should be 1 or 0 depending on whether $\boldsymbol{x}$ and $\boldsymbol{y}$ maps to 1 or 0.

Now notice in Equation [7] that the consequence in one implication is the premise in the next implication. And if we let $\boldsymbol{x}$ = (x ∈ {x}), for each x, then the proposition, $\boldsymbol{x}$, can now represent either the element or the set. So $\boldsymbol{x}$ can either be a premise or conclusion, whatever the implications require in Equation [7]. I labeled $\boldsymbol{\delta}_x(A)$ above as $\boldsymbol{\delta}_{yx}$ to remind us that A={y}. I call $\boldsymbol{\delta}_{yx}$ the point-to-point Dirac measure. It's not the Kronecker delta function because the input for $\boldsymbol{\delta}_{yx}$ is still elements, not numbers as required by the Kronecker delta.

Of course, for larger sets with more elements, these can be equated to the union of sets, each consisting of one element of the larger set. For example, if A={a,b,c,d,x,e,f,g,h}, then A={a}∪{b}∪{c}∪{d}∪{x}∪{e}∪{f}∪{g}∪{h}. Then we can write,

$$\boldsymbol{\delta}_{\mathrm{x}}(\mathrm{A}) \;=\; \sum_{\mathrm{y}\in\mathrm{A}} \boldsymbol{\delta}_{\mathrm{yx}} \;=\; \begin{matrix} 1 & \text{if} & \mathrm{x}\in\mathrm{A} \\ 0 & \text{if} & \mathrm{x}\notin\mathrm{A} \end{matrix}$$

The $\sum$ symbol is the symbol for summation; it means add the sum of the following terms. The y ∈ A under the $\sum$ symbol means that you add a term for each element in the set, A. Here y acts like a variable that takes on the value of each element in A, one at a time, until a term for all elements in A is added. For each value of y, the value of $\boldsymbol{\delta}_{yx}$ is determined and added to the sum. Eventually, y will equal x, if x ∈ A, and then $\boldsymbol{\delta}_{yx}$ will equal 1 for that term. All the rest of the terms will be 0. So the total sum will be 1. Note that y is the only thing varying, and since x is being held constant, $\boldsymbol{\delta}_{yx}$ can be treated as a function of the one variable element, y.

More generally, instead of using x and y, q_i and q_j will be used to label the elements. The corresponding propositions are $\boldsymbol{q}_i$ and $\boldsymbol{q}_j$, whose truth-value are determined by whether they are elements of the set, A, or not. But the implication between them, ($\boldsymbol{q}_i \rightarrow \boldsymbol{q}_j$), is determined by whether $q_j \in \{q_i\}$. And $q_i \in$A is determined by whether $1 \leq i \leq n$; if i is outside this range, then $q_i \notin$A, and $\boldsymbol{q}_i$ is false. This allows the use of the point-to-point Dirac measure, $\boldsymbol{\delta}_{q_i q_j}$, which becomes a function of one variable, i, since j is held constant.

Now since i and j take on whole number values, we can plot them along a number line. This number line can be seen as a coordinate system. The set, A, becomes a region along this line from 1 to n. Then the point-to-point Dirac measure, $\boldsymbol{\delta}_{q_i q_j}$, becomes a function of coordinates, $\delta(i,j)$, where j is held constant. In the case that i and j are discrete whole numbers, $\delta(i,j)$ is usually labeled as a Kronecker delta, δ_{ij}, where δ_{ij} = 1 only if $i = j$, and is otherwise 0, Note the use of the italic not bold font for the delta to indicate that it is a function of coordinates or indices, not elements. And using the Kronecker delta, that last equation can be written as,

$$\sum_{i=1}^{n} \delta_{ij} \;=\; \begin{matrix} 1 & \text{if} & 1 \leq j \leq n \\ 0 & \text{if} & \text{not} \end{matrix} \qquad [8]$$

And with this notation, we are now in a position to develop mathematical operations for conjunction and disjunction. The rest of this article is basically only concerned with the algebra.

THE MATH OF IMPLICATIONS

We've already mapped $\boldsymbol{T}$ to number 1 and $\boldsymbol{F}$ to number 0. But somehow we need to map AND and OR to some sort of math in order to construct a sensible math statement from a logic statement. To that end, consider the following,

$$\boldsymbol{A} \longrightarrow \boldsymbol{q}_j = \vee_{i=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \begin{cases} \boldsymbol{T}, \text{ since } \mathrm{q}_j \in \mathrm{A}, & \text{for } 1 \le j \le n \\ \boldsymbol{F}, \text{ since } \mathrm{q}_j \notin \mathrm{A}, & \text{for } j < 1 \text{ or } j > n\,. \end{cases} \qquad [9]$$

In the equation above, $\boldsymbol{A} \rightarrow \boldsymbol{q}_j$ is true if $\mathrm{q}_j \in \mathrm{A}$, since then $\boldsymbol{q}_j = (\mathrm{q}_j \in \mathrm{A})$ will be true, making the implication true. But $\boldsymbol{A} \rightarrow \boldsymbol{q}_j$ will be false if $\mathrm{q}_j \notin \mathrm{A}$. For then $\boldsymbol{q}_j$ will be false, But $\boldsymbol{A}$ will be true, since $\boldsymbol{A}$ here is defined as the expansion of all those elements that do exist in the set A. And likewise, $\vee^{n}_{i=1} (\boldsymbol{q}_i \rightarrow \boldsymbol{q}_j)$ is true if $\mathrm{q}_j \in \mathrm{A}$. For $(\boldsymbol{q}_i \rightarrow \boldsymbol{q}_j)$ means $\mathrm{q}_j \in \{\mathrm{q}_i\}$. And $\vee^{n}_{i=1}$ means i is in the range from 1 to n. So when j is in that range too, then there will be an i for which $i = j$, and there will be one term in the disjunction for which $\mathrm{q}_j \in \{\mathrm{q}_j\}$, which is true, making the whole disjunction true. But if $\mathrm{q}_j \notin \mathrm{A}$, then no such true term exists, all terms are false, making the whole disjunction false.

And when the Dirac measure is used to map Equation [9] to the math, it becomes Equation [8], repeated below.

$$\boldsymbol{\delta}_{\mathrm{q}_j}(\mathrm{A}) = \sum\nolimits_{i=1}^{n} \delta_{ij} = \begin{cases} 1 & \text{if } 1 \le j \le n \\ 0 & \text{if not} \end{cases} \qquad [8]$$

In Equation [8] above I just assumed that disjunction, ∨, is mathematically represented by addition. This is mostly to match the $\boldsymbol{T}$ or $\boldsymbol{F}$ of the implications to the 1 or 0 of the Dirac measures, both inside and outside the sum. But is there any way of proving this? And what math operation would we use for conjunction, ∧? For, in order to arrive at the Feynman Path Integral, all we need is a mathematical representation for conjunction, disjunction and implication. Since we already have that logic statements map to numeric values, 1 and 0, we must have logic operators map to math operators. Otherwise, the logic formulas would not translate into valid math formulas. And the primitive math operators that act on 1 and 0 are addition, subtraction, multiplication, and division, +, -, ×, and /. Yet we also need operators that commute in logic to map to operators that commute in math. For this will maintain the equality in both logic and math if the variable values should be interchanged. Since disjunction, ∨, and conjunction, ∧, are commutative, we are left to consider addition and multiplication. For $(a+b)=(b+a)$ and $(a\times b)=(b\times a)$, but $(a-b) \neq (b-a)$ and $(a/b) \neq (b/a)$. So to find the math operation for disjunction, ∨, we can consider the disjunction of Equation [9] with $n = 2$,

$$(\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_j) \vee (\boldsymbol{q}_2 \longrightarrow \boldsymbol{q}_j) = \begin{cases} \boldsymbol{T}, & \text{if } \mathrm{q}_j \in \mathrm{A}, \text{ or } 1 \le j \le 2 \\ \boldsymbol{F}, & \text{if not} \end{cases}$$

What we have so far is $\boldsymbol{T} \mapsto 1$, $\boldsymbol{F} \mapsto 0$, and $(\boldsymbol{q}_\mathrm{i} \rightarrow \boldsymbol{q}_\mathrm{j}) \mapsto \delta_{ij}$. So let's map ∨ to some as yet unknown math operation, call it ⊕ for now. Then the last equation above gets mapped to,

$$\delta_{1j} \oplus \delta_{2j} = \begin{cases} 1 & \text{if } 1 \le j \le 2 \\ 0 & \text{if } j < 1 \text{ or } j > 2 \end{cases}$$

Now if $j < 1$ or $j > 2$, then $\delta_{1j} = 0$, $\delta_{2j} = 0$, and $\delta_{1j} \oplus \delta_{2j} = 0$. But if $j = 1$, then $\delta_{1j} = 1$, $\delta_{2j} = 0$, and $\delta_{1j} \oplus \delta_{2j} = 1$. And if $j = 2$, then $\delta_{1j} = 0$, $\delta_{2j} = 1$, and $\delta_{1j} \oplus \delta_{2j} = 1$. But it's never the case that both δ_{1j} and δ_{2j} are 1 at the same time. So we have the following table,

Table 1

$(\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_j)$	$(\boldsymbol{q}_2 \longrightarrow \boldsymbol{q}_j)$	δ_{1j}	δ_{2j}	$(\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_j) \vee (\boldsymbol{q}_2 \longrightarrow \boldsymbol{q}_j)$	$\delta_{1j} \oplus \delta_{2j}$	condition
$\boldsymbol{F}$	$\boldsymbol{F}$	0	0	$\boldsymbol{F}$	0	$j < 1$ or $2 < j$
$\boldsymbol{F}$	$\boldsymbol{T}$	0	1	$\boldsymbol{T}$	1	$j = 2$
$\boldsymbol{T}$	$\boldsymbol{F}$	1	0	$\boldsymbol{T}$	1	$j = 1$

And the math operation that gives $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, and $1 \oplus 0 = 1$ would be addition, +, as originally suspected. It cannot be multiplication since there is a 0 for every condition, and anything times 0 is 0, and we'd never have a 1 as needed. So the mathematical map for Equation [9] is

$$\{ \quad \bigvee_{i=1}^{n} (\boldsymbol{q}_\mathrm{i} \longrightarrow \boldsymbol{q}_\mathrm{j}) = \boldsymbol{T}, \qquad [9] \quad \} \quad \mapsto \quad \{ \quad \sum_{i=1}^{n} \delta_{\mathrm{ij}} = 1, \qquad [8] \quad \}.$$

Next, let's find a math operator for conjunction, ∧. Equation [6] can be rewritten in the form,

$$(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \vee_{k=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_k) \wedge (\boldsymbol{q}_k \longrightarrow \boldsymbol{q}_j)\ ,$$

since it is arbitrary how we label the indices or what numbers we use to count them. Then we can consider the possibility of whether i or j or k is or is not in the range from 1 to n. If i, j, and k are all within that range, the truth of this equation is easy to understand. For k will cycle from 1 to n. When $k = i$, there will be a term of the form, $(\boldsymbol{q}_i \to \boldsymbol{q}_i) \wedge (\boldsymbol{q}_i \to \boldsymbol{q}_j) = (\boldsymbol{q}_i \to \boldsymbol{q}_j)$. And when $k = j$, there will be a term of the form, $(\boldsymbol{q}_i \to \boldsymbol{q}_j) \wedge (\boldsymbol{q}_j \to \boldsymbol{q}_j) = (\boldsymbol{q}_i \to \boldsymbol{q}_j)$. Since these two terms are the same, the disjunction of them in [6] is just $(\boldsymbol{q}_i \to \boldsymbol{q}_j)$.

When the Dirac measure is used, we map $(\boldsymbol{q}_i \to \boldsymbol{q}_j) \mapsto \delta_{ij}$, and we map $(\boldsymbol{q}_i \to \boldsymbol{q}_k) \mapsto \delta_{ik}$ and $(\boldsymbol{q}_k \to \boldsymbol{q}_j) \mapsto \delta_{kj}$ to get, $\delta_{ij} = \sum_{k=1}^{n} \delta_{ik} \odot \delta_{kj}$, where $\odot$ is the as yet unknown math operation for conjunction. Of course, this equation should be 1 only when $i = j$.

For $n = 1$, the value of k only goes to 1, and we have,

$$\delta_{ij} \;=\; \delta_{i1} \odot \delta_{1j}\ ,$$

and we can consider whether i or j is equal to 1 or not. And we have the following table for the math operation of conjunction,

Table 2

$(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_1)$	$(\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_j)$	δ_{i1}	δ_{1j}	$(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_1) \wedge (\boldsymbol{q}_1 \longrightarrow \boldsymbol{q}_j)$	$\delta_{ij} = \delta_{i1} \odot \delta_{1j}$	condition
$\boldsymbol{F}$	$\boldsymbol{F}$	0	0	$\boldsymbol{F}$	0	$i \neq 1, j \neq 1$
$\boldsymbol{F}$	$\boldsymbol{T}$	0	1	$\boldsymbol{F}$	0	$i \neq 1, j = 1$
$\boldsymbol{T}$	$\boldsymbol{F}$	1	0	$\boldsymbol{F}$	0	$i = 1, j \neq 1$
$\boldsymbol{T}$	$\boldsymbol{T}$	1	1	$\boldsymbol{T}$	1	$i = 1, j = 1$

From Table 2, the math operator, ⨀, must fulfill the requirement that 0⨀0 = 0, 0⨀1 = 0, 1⨀0 = 0, and 1⨀1 = 1. Clearly, ⨀ must be multiplication, ×, so that we have the map,

$$\{ \ (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \bigvee_{k=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_k) \wedge (\boldsymbol{q}_k \longrightarrow \boldsymbol{q}_j) \quad [6] \ \} \ \longmapsto \ \{ \ \delta_{ij} = \sum\nolimits_{k=1}^{n} \delta_{ik} \times \delta_{kj} \ \} \qquad [10]$$

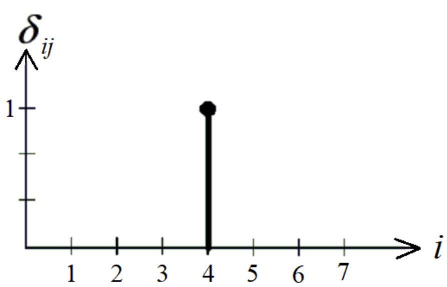
So far we have developed math for discrete values. But these formulas will prove most useful when using variables that could take on any value in a continuous range. In that case the summation sign, $\sum$, will become the integral of calculus, $\int$. The next section gives a brief introduction to integration.

INTEGRAL CALCULUS

This section is a brief introduction to the definition of integration as studied in calculus. If you are already familiar with calculus, you can skip to the next section. Or, here's a short video introduction to the integral, (www.youtube.com/watch?v=Stbc1E5t5E4).

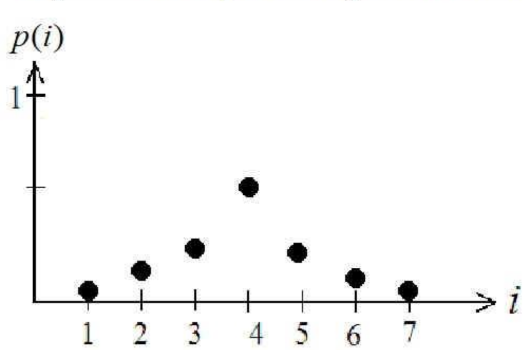
If we were to graph the Kronecker delta function, δ_{ij}, the value of i would be plotted along the horizontal axis and the numeric value of δ_{ij} would be plotted on the vertical axis as shown in Fig 1 below. Here, $j = 4$, and is held constant. Then the graph shows that when $i = j = 4$, then $\delta_{ij} = 1$, but is 0 for every other value of i.

Fig 1: Kronecker delta function



And a more general version of a discrete probability distribution might look like that in Fig 2 below, where the probability of the i^{th} alternative is labeled $p(i)$.

Fig 2: discrete probability distribution



Notice that all the points are well below 1 since we need the sum of all the values for the probability distribution to equal 1,

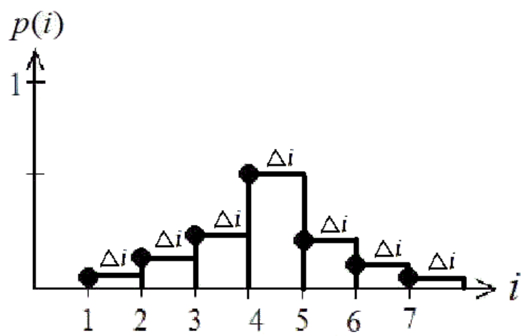
$$\sum_{i=1}^{n} p(i) = 1. \qquad [11]$$

But Equation [11] can also be written as,

$$1 = \sum_{i=1}^{n} (p(i) \times \Delta_i) \qquad [12]$$

where $\Delta_i = 1$ for all i. Equation [12] can be seen as a sum of areas each with a width of Δ_i and a height of $p(i)$ at various i, as is shown in Fig 3 below.

Fig 3: Area under distribution



The total area after summing these up is an approximation of the area between the i-axis and the curve represented by the function $p(i)$ from $i_{min} = 1$ to $i_{max} = 7$. More generally, however, we can make $\Delta_i = (i_{max} - i_{min}) / (n - 1)$, where in Fig 3, $i_{min} = 1$, $i_{\mathrm{max}} = 7$, and $n = 7$, so that $\Delta_i = (7 - 1) / (7 - 1) = 1$. When i takes on successive whole numbers on the i-axis, Δ_i will always be 1 and is usually omitted.

However, what happens when we want to divide the interval, $i_{min} \le i \le i_{max}$, by a larger number of sub-intervals? This would give us a closer approximation to the area under the $p(i)$ curve. In that case, Equation [12] can be written as

$$1 = \sum_{i=1}^{n} p(i_{min} + [i - 1] \Delta_i) \times \Delta_i \qquad [13]$$

Here n does not necessarily represent the number of whole number steps from i_{min} to i_{max} as before. The number n could be very large in which case $\Delta_i = (i_{max} - i_{min})/n$ and can become arbitrarily small as n increases. As i steps from 1 to n, $p(i_{min} + [i-1] \Delta_i)$ is evaluated in increments of Δ_i along the i-axis. With arbitrarily large values of n, $p(i)$ could be evaluated at any real value of i, not just whole numbers. And $p(i)$ will have to be a continuous function with a corresponding value for every real number of i for which $p(i)$ is evaluated.

So we must consider what happens as we let the discrete variable i become a continuous variable. When i become continuous, it's customary to label the i-axis as the x-axis, where x can take on any real value. Then $p(i)$ becomes $p(x)$ and must be a continuous function. The interval, i_{min} to i_{max}, becomes x_{min} to x_{max}, and Δ_i becomes $\Delta_x = (x_{max} - x_{min}) / (n - 1)$, and $i_{min} + [i - 1] \Delta_i$ becomes $x_j = x_{min} + [j - 1] \Delta_x$, where j still takes on values from 1 to n.

The process of integration found in the study of calculus is to let n increase without bound in Equation [13]. We say "in the limit as n approaches infinity" and write $\lim_{n\to\infty}$ in formulae and more simply $n \to \infty$ in text. And so the process of integration applied to Equation [13] would be written,

$$\lim_{n\to\infty} \sum_{j=1}^{n} p(x_j)\, \Delta_x = 1 \qquad [14]$$

Since $\Delta_x = (x_{max} - x_{min}) / (n - 1)$, then as n approaches infinity, $n \to \infty$, we have that Δ_x approaches zero, $\Delta_x \to 0$. But n never actually reaches infinity since that number is really not defined. And so Δ_x never actually reaches zero, but it is increasingly small. The notation of $\Delta_x \to 0$ is usually shortened to "dx" and is referred to as "differential x" meaning that it is increasingly small. And the function $p(x)$ in Equation [14] no longer assigns a probability for each discrete alternative as $p(i)$ did in Equation [11]. In Equation [14], $p(x)$ is a probability density, assigning a probability for events to happen between x and $x+dx$. The notation $\lim_{n\to\infty} \sum_{j=1}^{n}$ is a little cumbersome to write, so it is usually shortened to $\int_{x_{min}}^{x_{max}}$, where x_{min} is called the lower limit of integration and x_{max} is called the upper limit of integration. So changing to this notation Equation [14] becomes

$$\int_{x_{min}}^{x_{max}} p(x)dx = 1 \qquad [15]$$

And it is call the integral of $p(x)$ from x_{min} to x_{max} that is set equal to 1.

LOGIC OF DIRAC DELTAS

Now let's convert the summation of Equation [8] to an integral. This becomes necessary when the density of propositions become so dense that there is a continuous distribution of them. Using the techniques of the previous section, the continuous version of Equation [8] becomes the integral

$$\int_R \delta(x - x_0)dx = \begin{cases} 1, & \text{if } x_0 \in R \\ 0, & \text{if } x_0 \notin R \end{cases} \qquad [16]$$

where R is some interval on the x-axis. The function, $\delta(x-x_0)$, is called the [Dirac delta function](en.wikipedia.org/wiki/Dirac_delta_function), (en.wikipedia.org/wiki/Dirac_delta_function), and it is the continuous version of the Kronecker delta function, δ_{ij}. The notation, $\int_R$, means evaluate the integral within the region, R, from x_{min} to x_{max}. And instead of labeling each proposition with a discrete, whole number, i, the density of propositions is so great that we must go to a continuous variable, x. So the notation, $x_0 \in R$, for the Dirac delta function replaces the previous notation, $1 \le j \le 2$, for the Kronecker delta. And Equation [16] is one of the defining equations for the Dirac delta function, $\delta(x-x_0)$. The Dirac delta must be defined so that Equation [16] remains true independent of the size of the region, R. Even if R is specified to be very, very small, this integral must still evaluate to 1. Therefore, $\delta(x-x_0)$ becomes very large at x_0 so that when integrating, the area under the curve for $\delta(x-x_0)$ is still equal to 1 for very small R. But $\delta(x-x_0)$ is very small for any $x \neq x_0$ so that the area under the curve does not get too large when R is large.

And so the Dirac delta function is defined such that $\delta(x-x_0) \to \infty$ at $x = x_0$, and $\delta(x-x_0) \to 0$ at $x \neq x_0$. Both these limiting processes of $\delta(x-x_0) \to \infty$ and $\delta(x-x_0) \to 0$ are controlled by a single parameter, Δ, I'll call it cap-delta. So as $\Delta \to 0$, you get $\delta(x-x_0) \to \infty$ for $x = x_0$ and you get $\delta(x-x_0) \to 0$ for $x \neq x_0$. This is a different limiting process than the $n \to \infty$ limit for integration. One has to hold Δ at some finite value and then do the integration on the continuous Dirac delta function, $\delta(x-x_0)$. And then after integration is done, the limiting process of $\Delta \to 0$ is done. For it would not be possible to do the integration if one were to allow $\delta(x-x_0)$ to approach infinity first. This is because ∞ times dx is not defined.

In the literature the region R in Equation [16] is usually the entire real line, $-\infty \leq x \leq +\infty$, but this does not necessarily have to be the case. Yet if R in Equation [16] were the entire real line, then x_0 would certainly be included in it, and we get,

$$\int_{-\infty}^{+\infty} \delta(x-x_0)dx \quad = \quad 1 \qquad , \qquad [17]$$

for any value of x_0. This is a map from the logical Equation [9] and the Kronecker delta Equation [8].

However, if R has upper and lower limits, $x_{min} \leq x \leq x_{max}$, then the integral of Equation [16] is 0 when $x_0 \notin R$. This is because x_0 is outside the limits of integration, so $\delta(x-x_0)$ is basically 0 everywhere it is integrated; so the integral is 0.

The intent here is to use the Dirac delta function to transform Equation [7] into the Feynman Path Integral. But Equation [7] was derived by iterating Equation [6]. So if the map developed so far, to go from logic to math is indeed valid, then the Dirac delta function should also have this same iterative property.

To that end, consider what effect Equation [16] would have on an arbitrary function $f(x)$,

$$\int_R f(x)\delta(x-x_0)dx \; .$$

Since the function $\delta(x-x_0)$ is practically 0 away from x_0 and very large at x_0, we have that $f(x)\,\delta(x-x_0)$ is practically 0 away from x_0 and large at x_0. This means we can restrict the interval of integration to a very small interval, R', that includes x_0. Then $x_0 \in R'$, and $R' \subset R$. And when R' becomes very small, $f(x)$ will essentially be $f(x_0)$ if R' is a small enough interval around x_0. Then the above equation becomes

$$\int_R f(x)\delta(x-x_0)dx \quad = \quad f(x_0)\int_{R'} \delta(x-x_0)dx \quad = \quad f(x_0)\cdot 1 \quad = \quad f(x_0) \; .$$

So that we have,

$$\int_R f(x)\delta(x-x_0)dx \quad = \quad f(x_0) \qquad \text{for } x_0 \in R \qquad [18]$$

But as usual, if $x_0 \notin R$, then $f(x_0)\ \delta(x-x_0)$ will essentially be 0 throughout the interval of integration, and

$$\int_R f(x)\delta(x-x_0)dx \;=\; 0 \qquad \text{for } x_0 \notin R. \tag{19}$$

Now x in these equations is called a dummy variable of integration, and we are free to change it to anything we like without changing the value of the integral. So let's change the integration variable from x to x_1. Then $f(x)$ becomes $f(x_1)$, and $\delta(x-x_0)$ becomes $\delta(x_1-x_0)$, and dx becomes dx_1, and Equation [18] becomes

$$\int_R f(x_1)\delta(x_1-x_0)dx_1 \;=\; f(x_0) \qquad \text{for } x_0 \notin R.$$

But if $f(x_1)$ were to be a Dirac delta function itself $\delta(x-x_1)$, we get

$$\int_R \delta(x-x_1)\delta(x_1-x_0)dx_1 \;=\; \begin{cases} \delta(x-x_0) & \text{for both } \{x, x_0\} \in R \\ 0 & \text{for either of } \{x, x_0\} \notin R \end{cases} \tag{20}$$

Of course, now both x and x_0 must be in the interval of integration, R. Otherwise, if $x \notin R$, then $\delta(x-x_1)$, would be 0 throughout the integration, making the integral 0. And if $x_0 \notin R$, then $\delta(x_1-x_0)$ would be 0 throughout the integration, making the integral 0.

Note that Equation [20] is a Dirac delta representation of Equation [6]. So the Dirac delta function has the same iterative property corresponding to its logical counterpart. This is what initially attracted me to the Dirac delta function as a math representation for implication. It took some work, however, to justify this using the Dirac measure.

Iterating again we get,

$$\int_R\int_R \delta(x-x_2)\delta(x_2-x_1)\delta(x_1-x_0)dx_2dx_1 = \int_R \delta(x-x_1)\delta(x_1-x_0)dx_1 = \delta(x-x_0)$$

$$\text{for } \{x, x_0\} \subseteq R \tag{21}$$

And iterating an infinite number of times we get,

$$\int_R\int_R\cdots\int_R \delta(x-x_n)\delta(x_n-x_{n-1})\cdots\delta(x_1-x_0)dx_ndx_{n-1}\cdots dx_1 = \delta(x-x_0)$$

$$\text{for } \{x, x_0\} \subseteq R \tag{22}$$

Obviously each of $x_1, x_2, \ldots, x_n$ is within the interval of R since we are integrating with respect to those variables within R. And note that Equation [22] can be seen as the Dirac delta representation of Equation [7]. The mathematical map developed so far seems consistent. And we are closer to the math of the path integral.

To sum up, the progression has been to go from logical equations to discrete summations to integrals,

$$\vee_{i=1}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \boldsymbol{T} \quad \mapsto \quad \sum_{i=1}^{n} \delta_{ij} = 1 \quad \mapsto \quad \int_R \delta(x - x_0) dx = 1 \, .$$

An iterative property also follows from logic to the Kronecker delta to the Dirac delta function,

$$(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \vee_{k=0}^{n} (\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_k) \wedge (\boldsymbol{q}_k \longrightarrow \boldsymbol{q}_j)$$

$$\mapsto \quad \delta_{ij} = \sum_{k=1}^{n} \delta_{ik} \cdot \delta_{kj} \quad \mapsto \quad \delta(x - x_0) = \int_R \delta(x - x_1) \delta(x_1 - x_0) dx_1 \, .$$

All that remains is to find a mathematical expression for $\delta(x_1 - x_0)$ that has these same properties. There may be many functions that could be used to represent the Dirac delta function. One such function is the Gaussian form of the Dirac delta,

$$\delta(x - x_0) = \lim_{\Delta \to 0} \frac{1}{(\pi \Delta^2)^{1/2}} \, e^{-(x - x_0)^2 / \Delta^2} \qquad [23]$$

It has the property that as Δ approaches zero, the delta function becomes infinite in such a way that the integral of Equation [16] remains one. The integration of the Gaussian Dirac delta is a little tricky to prove and is done in many books on quantum mechanics that cover the path integral. (No physics is necessary in the proof.) Here's a video of how to integrate a Gaussian function, (www.youtube.com/watch?v=fWOGfzC3IeY). The Gaussian Dirac delta function of Equation [23] also satisfies the iterative property of Equation [20] since,

$$\left(\frac{\lambda}{2\pi(t - t_0)} \right)^{1/2} e^{\frac{\lambda (x - x_0)^2}{2(t - t_0)}} = \int_{-\infty}^{+\infty} \left(\frac{\lambda}{2\pi(t - t_1)} \right)^{1/2} e^{\frac{\lambda (x - x_1)^2}{2(t - t_1)}} \left(\frac{\lambda}{2\pi(t_1 - t_0)} \right)^{1/2} e^{\frac{\lambda (x_1 - x_0)^2}{2(t_1 - t_0)}} dx_1 \qquad [24]$$

where $(t - t_1)$ and $(t_1 - t_0)$ both act like the previous Δ and approach zero as $(t - t_0)$ approaches zero. This equation is called a Chapman-Kolmogorov , (mathworld.wolfram.com/Chapman-KolmogorovEquation.html), equation and is used in The Feynman Integral and Feynman's Operational Calculus, by Gerald W. Johnson and Michael L. Lapidus, page 37. I've not found any other function that satisfies this iteration property other than the exponential Gaussian function. And it's fortunate that this integral solves the iteration property exactly. For we need to solve the integral before we can allow the parameter to approach zero as required by the Dirac delta function.

But if the exponential Gaussian is to be used for the Dirac delta function, then notice in Equation [23] this makes $\delta(x_1 - x_2) = \delta(x_2 - x_1)$, since in the exponent, $(x_1 - x_2)^2 = (x_2 - x_1)^2$. Yet, x_1 is stated by proposition, $\boldsymbol{p}_1$, to be the position of the element, p_1, and x_2 is stated by $\boldsymbol{p}_2$ to be the position of p_2. And we know that for material implication, $(\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_2) \neq (\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_1)$. So we need to modify Equation [23] to prevent the equality when the coordinates are interchanged. The only parameter left to manipulate in Equation [23] is Δ. We need to have Δ depend on whether we use $(x_1 - x_2)$ or $(x_2 - x_1)$ in the exponent of the Gaussian function.

Let's start with the simple substitution $\Delta^2 = (t_1 - t_2)$ in Equation [23]. Here we are letting successive values of t mark off successive steps along a path. So if $(t_1 - t_2)$ marks off the path, $(\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_2)$, then $(t_2 - t_1)$ marks off the reverse path, $(\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_1)$. And if $t_2 > t_1$, then the exponent in Equation [23] will be positive in the $(t_1 - t_2)$ direction but negative in the reverse $(t_2 - t_1)$ direction. And we will have $\delta(x_1 - x_2) \neq \delta(x_2 - x_1)$ as required since $(\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_2) \neq (\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_1)$.

But since $\Delta^2 = (t_1 - t_2)$, then as $\Delta^2 \rightarrow 0$, to form a Dirac delta, we will get $(t_1 - t_2) \rightarrow 0$. And in the $(t_1 - t_2)$ direction, the exponential will approach infinity as $(t_1 - t_2) \rightarrow 0$ since $(t_1 - t_2) < 0$ in that case. But in the reverse $(t_2 - t_1)$ direction, the exponential will approach zero as $(t_2 - t_1) \rightarrow 0$ since $(t_2 - t_1) > 0$ in that case. This would seem to make the delta representation for $(\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_2)$ to be of a very different character than $(\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_1)$, the first mapping to infinity and the second mapping to zero.

Not only this, but in the $(t_1 - t_2)$ direction, the leading factor will be a complex number since it is taking the square root of a negative number; $(t_1 - t_2) < 0$. But in the reverse $(t_2 - t_1)$ direction, the leading factor will be a real number since it is taking the square root of a positive number; $(t_2 - t_1) > 0$. Now we have a totally different character for one direction than another, one way is complex, the other way is real.

But it was totally arbitrary to assign which direction through a path got greater values of t. If we were to have assigned greater values of t in the opposite direction, then the first direction would have been real and the second complex. Also, it's possible to construct paths that wind about in such strange ways that $(\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_2)$ could be in the forward direction in some paths and in the reverse direction in other paths. As you construct every possible path from start to finish, each step is used many different times, sometimes in the forward direction and sometimes in the reverse direction. So we don't want to have steps that greatly differ in character depending on which way you walk through them. And we don't want to give preferential treatment to any step or group of steps. We want all the steps to have the same character and have equal importance. For any one of them could have just as easily been in the start or middle or end of a path.

This can be done if we modify Δ^2 and make it $\Delta^2 = i(t_1 - t_2)$, where $i = \sqrt{-1}$. Then the only difference between $\delta(x_1 - x_2)$ = and $\delta(x_2 - x_1)$ is a phase shift. One is the complex conjugate of the other. They are both complex, and they both have the same absolute value. And our delta function becomes,

$$\delta(x_j - x_k) = \lim_{t_j \to t_k} \left(\frac{1}{\pi i (t_j - t_k)} \right)^{1/2} e^{\frac{-(x_j - x_k)^2}{i(t_j - t_k)}} \qquad [25]$$

PATH INTEGRATION

So let us make the following substitution in Equation [23],

$$\Delta^2 = \frac{2i\hbar}{m}(t - t_0) \quad , \qquad [26]$$

where m and $\hbar$ are arbitrary constants for the purposes here, and $i = \sqrt{-1}$. Then we can rearrange Equation [23] to get,

$$\delta(x - x_0) = \lim_{t \to t_0} \left[\frac{m}{2\pi i\hbar(t - t_0)} \right]^{1/2} \exp\left[\frac{im(x - x_0)^2}{2\hbar(t - t_0)} \right] = \lim_{t \to t_0} \left[\frac{m}{2\pi i\hbar(t - t_0)} \right]^{1/2} \exp\left[\frac{im}{2\hbar} \left(\frac{x - x_0}{t - t_0} \right)^2 (t - t_0) \right]$$

which equals

$$\lim_{\Delta t \to 0} \left[\frac{m}{2\pi i\hbar\Delta t} \right]^{1/2} \exp\left[\frac{im}{2\hbar} \left(\frac{\Delta x}{\Delta t} \right)^2 \Delta t \right] = \lim_{\Delta t \to 0} \left[\frac{m}{2\pi i\hbar\Delta t} \right]^{1/2} e^{\left(\frac{im}{2\hbar} (\dot{x})^2 \Delta t \right)} \quad , \qquad [27]$$

where $\Delta x = (x-x_0)$ and $\Delta t = (t-t_0)$, and where $\dot{x} = \Delta x/\Delta t = (x-x_0)/(t-t_0)$. Using m and $\hbar$ above is not an attempt to covertly introduce physics. Here m and $\hbar$ are constants of proportionality. It is only fortunate that they appear to be mass and Planck's constant.

And inserting Equation [27] into Equation [22], we get

[28]

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2} e^{\left(\frac{im}{2\hbar}(\dot{x}_n)^2\Delta t\right)}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2} e^{\left(\frac{im}{2\hbar}(\dot{x}_{n-1})^2\Delta t\right)}\cdots\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2} e^{\left(\frac{im}{2\hbar}(\dot{x}_1)^2\Delta t\right)} dx_n\cdots dx_1$$

with the appropriate limits implied, and where the R in the integrals of Equation [22] is the entire real line. Since each of $t_1,t_2,t_3,...,t_n$, is between t_0 and t, then as $(t-t_0)\rightarrow 0$ for $\delta(x-x_0)$, then every $(t_j-t_k)\rightarrow 0$ for every $\delta(x_j-x_k)$ in Equation [22]. This is why I simply write Δt instead of t_j-t_k in Equations [28]. This makes all the square root factors all the same. And we can multiply all n of the $(m/2\pi i\hbar\Delta t)^{1/2}$ factors together to get $(m/2\pi i\hbar\Delta t)^{n/2}$ 2 . And Equation [28] becomes,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\frac{i}{\hbar}\sum_{j=0}^{\infty}\frac{m}{2}(\dot{x}(t_j))^2\Delta t} dx_1 dx_2\cdots dx_n$$

[29]

This is because the exponents add in Equation [28], and there is n of them, one for each step in the path. So for any one path, j starts from 0, the starting point, to n, the ending point. And we add them all up, $\sum^{n}_{j=0}$. And as n increases without bound, there is an infinite number of t's between the start and end of the path, so the difference between adjacent t's approaches 0. Or in other words, $(t_j-t_k)=\Delta t\rightarrow 0$. This turns the summation in Equation [29] into an integral, and we get,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\frac{i}{\hbar}\int_{0}^{t}\frac{m}{2}(\dot{x})^2 dt} dx_1 dx_2\cdots dx_n.$$

[30]

Notice that the exponential factor looks like the Action integral for a particle in motion without any force applied. Here m and $\hbar$ are only constants of proportionality. Remember that the increasing direction and scaling size of t was arbitrary, t was an arbitrary parameterization of paths. And which coordinate system to use, the positive direction and scaling of x is also arbitrary. So m and $\hbar$ work together to cancel out whatever units occur in the integral. This is necessary because the exponent must be a pure number without units in order to evaluate it, an exponent with units attached, like feet or seconds, doesn't make sense. Equation [30] can be recognized as the Feynman Path Integral for the propagator of the wave function for a free particle in quantum mechanics. The limits of the Δt approaching zero is understood by the notation of dt and Δt. But this formula was derived by considering how any two points in space are connected through paths connecting every other point in space. It shows how all of space is connected. Yet, if this formula is about space, then how can it be about a particle which travels through space? After all, particles are different than space, right? Yet, even with a particle, there is a starting point and an ending point in its trajectory. And if there is no means to determine intermediate points in its trajectory, then you are left to consider every possible path it might have taken. So Equation [30] duplicates the math for only the kinetic energy of a particle, but what logic might account for the potential energy of a particle? The next section addresses this.

THE POTENTIAL IMPLICATIONS

Now what if there were something in space which determined that implications will be stronger or weaker at various places? Then each of the implications in a path will be weighted by a function, $\rho(x)$? The greater the value of this function, then the more or less effect an implication would have in a path. The function, $\rho(x)$, would strengthen the effect of an implication or strengthen it in the opposite direction. In the math, $\rho(x)$ would be a factor that would be capable of changing $\delta(x_j - x_i)$ into its complex conjugate since the complex conjugate of an implication is an implication in the opposite direction. So $\rho(x)$? itself would be a complex number. And instead of $\delta(x_j - x_i)$, you would have $\rho(x) \cdot \delta(x_j - x_i)$. Then Equation [22] becomes,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \rho(x_n)\delta(x - x_n)\rho(x_{n-1})\delta(x_n - x_{n-1}) \cdots \rho(x_0)\delta(x_1 - x_0)dx_n \cdots dx_1 \quad [31]$$

And Equation [27] becomes

$$\lim_{\Delta t \to 0} \left[\frac{m}{2\pi i\hbar\Delta t}\right]^{1/2} \rho(x) e^{\left(\frac{im}{2\hbar}(\dot{x})^2 \Delta t\right)} \quad [32]$$

But since $\rho(x)$ is a complex number, we can write $\rho(x) = e^{-iV(x)\Delta t/\hbar}$, where $-\infty < V(x) < +\infty$. Then Equation [32] becomes

$$\lim_{\Delta t \to 0} \left[\frac{m}{2\pi i\hbar\Delta t}\right]^{1/2} e^{i\left(m(\dot{x})^2/2 \,-\, V(x)\right)(\Delta t/\hbar)} . \quad [33]$$

Note, however, that $\rho(x)$ is never 0 or ∞ . Again, this is so we don't create any great differences in character between one implication and another. We don't want to give any preferential treatment to any arbitrary points of space. So the only thing that $\rho(x)$ can do is introduce a phase shift and change the angle and possibly reverse its direction. If $V(x)$ were of the right value at x, then the exponent, $m(\dot{x})^2/2 - V(x)$, becomes negative and turns $\delta(x_j - x_i)$ into something closer to its complex conjugate so that it acts more like a step in the opposite direction. But $V(x)$ does not change the magnitude of a step, since $||e^{-iV(x)}|| = 1$. The result of $V(x)$ on every possible path is to act like a potential, changing the overall path from what it would be without it.

With $V(x)$ added, Equation [30] becomes

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\frac{i}{\hbar}\int_0^t (m(\dot{x})^2/2 \,-\, V(x))dt} dx_1 dx_2 \cdots dx_n \quad [34]$$

This is the Feynman Path Integral for a particle in a potential. It is the wave function of a particle, $\psi(x,t)$. But what could cause a potential to occur? I have a little bit more explanation here, (logictophysics.com/virtual.html).

THE BORN RULE OF PROBABILITIES

The Born rule tell us that the probability density, $p(x)$, for finding a particle between x and $x+dx$ that has wave function, $\psi(x,t)$, is equal to the wave function times the complex conjugate of the wave function. Or in symbols,

$$p(x) = |\psi(x,t)|^2 = \psi(x,t) \cdot \overline{\psi}(x,t) \ .$$

This can be explained with the formulism developed here. Equation [4] is

$$\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \rightarrow (\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2) \wedge (\boldsymbol{q}_2 \rightarrow \boldsymbol{q}_1)$$

which is an equality if at least one of $\boldsymbol{q}_1$ or $\boldsymbol{q}_2$ is true. When this is mapped in mathematical terms, each of $\boldsymbol{q}_1$ or $\boldsymbol{q}_2$ is a proposition mapped to a value between 0 and 1 depending on how likely it is. So, for example, $\boldsymbol{q}_1$ maps to a number that behaves as the probability that the proposition $\boldsymbol{q}_1$ is true. And factors like $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2)$ generate the path integral which is another way of describing the wave function, $\psi\,(x,t)$. We learned that $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2)$ maps to a complex number, and $(\boldsymbol{q}_2 \rightarrow \boldsymbol{q}_1)$ maps to its complex conjugate. Then conjunction, $\wedge$, maps to multiplication. So $\boldsymbol{q}_1 \wedge \boldsymbol{q}_2$ maps to a probability of finding $\boldsymbol{q}_1$ time the probability of finding $\boldsymbol{q}_2$, or $p(\boldsymbol{q}_1) \cdot p(\boldsymbol{q}_2)$.

The physical interpretation of $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2)$ is that the state described by a proposition $\boldsymbol{q}_1$ leads to the state described by proposition $\boldsymbol{q}_2$. In terms of an experiment, $\boldsymbol{q}_1$ would be the setup of the experiment and $\boldsymbol{q}_2$ would be the measured result. Now, experiments are set up in a known state with certainty so that the results can be repeated. That means here that $p(\boldsymbol{q}_1)$ would be 1 by deliberate design. So what we have left is $p(\boldsymbol{q}_2)$ equal to a wave function representing $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2)$ times the complex conjugate of the wave function representing $(\boldsymbol{q}_2 \rightarrow \boldsymbol{q}_1)$. If we let $\boldsymbol{q}_2$ be located at x, then $p(\boldsymbol{q}_2)$ is replaced by $p(x)$, and $(\boldsymbol{q}_1 \rightarrow \boldsymbol{q}_2)$ is represented by $\psi(x,t)$, and $(\boldsymbol{q}_2 \rightarrow \boldsymbol{q}_1)$ is represented by $\overline{\psi}(x,t)$. And so we get the Born rule:

$$p(x) = \psi(x,t) \cdot \overline{\psi}(x,t) = |\psi(x,t)|^2 \ , \qquad [35]$$

where $\psi(x,t)$ must be interpreted as the square root of a probability.

The wave function expresses how one fact implies another. But it does not give enough information to predict the probabilities of a measurement. This is because an implication could be true independent of the premises. An implication does not give us information about the premise. But an experiment is specified by both the premise and conclusion, by both the setup and a measurement. In order to form a workable hypothesis with repeatable results, both the setup and measurement apparatus must be fully specified. You must know that the setup and the result both exist in conjunction. Otherwise you cannot form a correlation between cause and effect if you don't know what caused your effect or if you don't know what effect your cause had. So the wave function tells us what effect a cause will have, and the conjugate wave function tells us what caused an effect. And together you know both cause and effect and you can calculate the relationship (probability) between them.

And it seems only intelligence is concerned with calculating the probability between cause and effect. A screen hit by an electron doesn't care where it came from; it could come from anywhere and have the same effect. And an atom emitting a photon doesn't care what effect the photon has on any screen. Physical events don't care what the probabilities are; they simply respond to stimuli. But conscious beings with intelligence calculate probabilities so they can make intelligent decisions. This is likely what is meant when scientists say that observation (from conscious beings) collapses the wave function to the measured result. It is only conscious beings that form correlations between proposed causes and effects.

THE LARGER IMPLICATIONS

The quantum mechanics of the wave function (or path integral) is usually called 1st quantization. Functions are obtained with this procedure. There is also a branch of quantum physics called quantum field theory which is sometimes called 2nd quantization. It takes the fields obtained in 1st quantization and plugs them into a very similar quantization procedure to get 2nd quantization. Again, it seems like there is little justification for further quantizing fields other than it just so happens to predict results. It occurs to me, however, that quantum field theory comes naturally to the procedure described here.

We started with the fact that

$$\wedge_{i=1}^{n}\boldsymbol{q}_i \longrightarrow \wedge_{ij=1}^{n}(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) \qquad [5]$$

which is an equality if at least one of the $\boldsymbol{q}_i$ is true. And so it became necessary to evaluate

$$(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) = \vee_{i_1,i_2,i_3,i_4,\ldots,i_m=0}^{n}(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_{i_1})\wedge(\boldsymbol{q}_{i_1} \longrightarrow \boldsymbol{q}_{i_2})\wedge\ldots\wedge(\boldsymbol{q}_{i_m} \longrightarrow \boldsymbol{q}_j) \qquad [7]$$

that when represented in mathematical form became the path integral of 1st quantization.

But there is no reason not to apply Equation [5] again to get

$$\wedge_{i=1}^{n}\boldsymbol{q}_i \longrightarrow \wedge_{ij=1}^{n}(\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) \longrightarrow \wedge_{ijkl=1}^{n}((\boldsymbol{q}_i \longrightarrow \boldsymbol{q}_j) \rightarrow (\boldsymbol{q}_k \longrightarrow \boldsymbol{q}_l)) \, ,$$

in which the last conjunction is an equality if at least one of the ($\boldsymbol{q}_i \rightarrow \boldsymbol{q}_j$) is true, which will be the case if at least one of the $\boldsymbol{q}_i$ is true. And if we let $\boldsymbol{q}_{ij} = (\boldsymbol{q}_i \rightarrow \boldsymbol{q}_j)$, then we have

$$\wedge_{i=1}^{n}\boldsymbol{q}_i \longrightarrow \wedge_{ijkl=1}^{n}(\boldsymbol{q}_{ij} \longrightarrow \boldsymbol{q}_{kl}) \, .$$

And likewise, this would necessitate the evaluation of

$$(\boldsymbol{q}_{ij} \longrightarrow \boldsymbol{q}_{kl}) = \vee_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m=1}^{n}(\boldsymbol{q}_{ij} \longrightarrow \boldsymbol{q}_{i_1j_1})\wedge(\boldsymbol{q}_{i_1j_1} \longrightarrow \boldsymbol{q}_{i_2j_2})\wedge\ldots\wedge(\boldsymbol{q}_{i_mj_n} \longrightarrow \boldsymbol{q}_{kl})$$

In this case the mathematical representation of ($\boldsymbol{q}_{i1j1} \rightarrow \boldsymbol{q}_{i2j2}$) would be

$$\delta(\varphi_{i_1j_1}(x,t) - \varphi_{i_2j_2}(x,t)) \, , \qquad [36]$$

where $\varphi_{i1j1}(x,t)$ is the wave function of 1st quantization and is the mathematical representation of $\boldsymbol{q}_{i1j1} = (\boldsymbol{q}_{i1} \rightarrow \boldsymbol{q}_{j1})$. The delta here would be expected to still be an exponential Gaussian with $\varphi_{i1j1}(x,t)$ replacing x_i in the exponent. And $d\varphi$ would replace dx in the integrals to finally get

$$\int\int\cdots\int\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\,i\int_0^t\int L(\dot{\varphi},\varphi,t)dxdt}\, d\varphi_1 d\varphi_2\cdots d\varphi_n \ , \qquad [37]$$

which is the path integral of 2nd quantization used in quantum field theory.

And I suppose the same procedure can be used to get 3rd quantization except that keeping track of the indices might be quite tedious.

Previously the complex numbers were used in the wave function of 1st quantization. And the complex numbers establish the U(1) symmetry of QED. I have to wonder if a similar effort for the four numbers associated with the ($\boldsymbol{q}_{i1j1}\rightarrow\boldsymbol{q}_{i2j2}$) of second quantization or the eight numbers associated with third quantization might establish the quaternions or octonions used in the quaternionic representation of Isospin, SU(2), or the octonionic formulation of SU(3) used in particle physics. I am by no means an expert in these matters. I only noticed their use in my reading, and now it seems they may become relevant. John Baez has a brief introduction to quaternions and octonions here, (www.math.niu.edu/~rusin/known-math/95/octonions.phys). There the iteration from complex numbers to quaternions to octonions is very similar to the iteration from first to second to third quantization here and suggests their use. Further references on quaternions and octonions in symmetry groups in physics are here, (www.amazon.com/Division-Algebras-Quaternions-Mathematics-Applications/dp/0792328906#_), and here, (benasque.org/2011qfext/talks_contr/2034_Bisht.pdf).

DISCLAIMER

Having noticed a parallel between paths constructed from logical implication and paths constructed of particle trajectories, I extended that analogy to reconstruct Feynman's Path Integral from simple logic. The conversion is achieved by representing the material implication of logic with the Dirac delta function and then using the complex Gaussian form of the Dirac delta. However, at this point my derivation has not been subjected to peer review. It has yet to pass inspection by mathematical logicians. Until that time, this effort should be considered preliminary.

I may not have given a full account of all of the quantum mechanical formulism yet. I've not derived Schrodinger's equation, eigenvalues and eigenvectors, Hilbert or Fock space, or Heisenberg's uncertainty principle, for example. But I suspect that the rest may be implied by the wave function that I have derived. For example, the Schrodinger equation is derived from the path integral in many quantum mechanics text.

Keep in mind that I'm not claiming to have derived all of physics from logic. In order to claim a logical derivation of physics, one would have to derive physical quantities such as some of the 20 or so constants of nature or the principles of General Relativity. So I will keep an eye on such efforts. And I'll try to include more as time and insight allow.

However, this does open an intriguing possibility for deriving the laws of nature. Typically physicists use trial and error methods for finding mathematics that describe the data of observation in very clever ways. These theories are then used to make predictions that experiment may confirm or falsify. When very many observations are consistent with the equations, we have confidence that the theory is correct. However, such theories can never be proven correct and are always contingent on future observations confirming them. But we can never say they are completely proven true. For we don't know whether some observation in the future may falsify the theory. Now, however, there may be the possibility that physical theory can be derived from logical considerations alone. Such a theory would in essence be a tautology and proved true by derivation. We would have to check our math against observation, of course. But if even one observation was consistent with such a theory, how could we say that other observations would not be? Can we expect that some parts of nature are logical but others are not when they coexist in the same universe?

We may not have any choice but to derive physics from logic since the ability to confirm ever deeper theories will require energies that are beyond our abilities to control. After all, we cannot recreate the universe from scratch many times over in order to confirm some proposed theory of everything. So we may be forced to rely on logical consistency alone. And I think I have a start in that direction.

Now, having derived the transition amplitudes of a particle from logic alone, I use these transition amplitudes in a description of virtual particle pairs. These virtual particle pairs come directly from the conjunction of points on a manifold and can be used to describe many of the phenomena we see in nature, perhaps all. See other posts in the TOC page at logictophysics.com.